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CHARACTER SUM ANALOGUES OF CONSTANT TERM IDENTITIES FOR ROOT SYSTEMS

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ABSTRACT

Certain identities connected with root systems provide explicit constant terms in Laurent series expansions of multivariable functions. Character sum analogues of these identities are given.

1. Introduction and notation

We use the following notation. Let p be an odd prime, and consider the finite field GF (q), where q is a power of p. The symbol Σ_x denotes summation over all $x \in \mathrm{GF}(q)$. Throughout, A, B, C denote arbitrary characters on $\mathrm{GF}(q)$, 1 denotes the trivial character, and ϕ is the quadratic character on $\mathrm{GF}(q)$. The multiplicative inverse of a nonzero element $x \in \mathrm{GF}(q)$ is written $\bar{x}$. Define the Gauss sum over $\mathrm{GF}(q)$ by

$$G(A)=\sum_x A(x)\zeta^{\mathrm{Tr}(x)},$$

where Tr is the trace map from $\mathrm{GF}(q)$ to $\mathrm{GF}(p)$ and $\zeta=\exp(2\pi i/p)$. Define the Jacobi sum over $\mathrm{GF}(q)$ by

$$J(A,B)=\sum_x A(x)B(1-x).$$

Each of the five nontrivial identities below gives the constant term (abbreviated C.T.) of a Laurent series in variables $x_1,\cdots,x_n$ expanded about the origin. We write $x_1=x$, $x_2=y$ in the case $n=2$, and we let a, b, c, $a_1,\cdots,a_n$ denote positive integer contants.

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$$\text{(1)} \qquad \text{C.T.} \prod_{1\leq i\neq j\leq n} (1-x_i x_j^{-1})^{a_i} = \frac{(a_1+\cdots+a_n)!}{a_1!\,a_2!\cdots a_n!},$$

$$\text{(2)} \quad \text{C.T.}\prod_{i=1}^{n}(1-x_i)^a(1-x_i^{-1})^b \prod_{1\leq i\neq j\leq n}(1-x_i x_j^{-1})^c = \prod_{j=0}^{n-1}\frac{(a+b+jc)!(jc+c)!}{(a+jc)!(b+jc)!c!},$$

$$\text{(3)} \qquad \begin{aligned} &\text{C.T.}(1-x)^a(1-x^{-1})^a(1-y)^b(1-y^{-1})^b(1-xy)^c(1-x^{-1}y^{-1})^c \\ &= \frac{(a+b+c)!(2a)!(2b)!(2c)!}{a!b!c!(a+b)!(a+c)!(b+c)!}, \end{aligned}$$

$$\text{(4)} \qquad \begin{aligned} &\text{C.T.}(1-x)^a(1-x^{-1})^a(1-y)^a(1-y^{-1})^a \\ &\times(1-xy^{-1})^b(1-x^{-1}y)^b(1-xy)^c(1-x^{-1}y^{-1})^c \\ &= \frac{(2a+b+c)!(2a)!(2b)!(2c)!}{(a+b+c)!(a+b)!(a+c)!a!b!c!}, \end{aligned}$$

$$\text{(5)} \qquad \begin{aligned} &\text{C.T.}\prod_{i=1}^{n}(1-x_i)^a(1-x_i^{-1})^a(1-x_i^2)^c(1-x_i^{-2})^c \cdot \\ &\cdot \prod_{1\leq i<j\leq n}(1-x_i x_j)^b(1-x_i^{-1}x_j^{-1})^b(1-x_i x_j^{-1})^b(1-x_j x_i^{-1})^b \\ &= \prod_{j=0}^{n-1}\frac{(2a+2c+2jb)!(2c+2jb)!(jb+b)!}{(a+2c+(n+j-1)b)!(a+c+jb)!(c+jb)!b!}. \end{aligned}$$

Formulas (1)–(5) are given in [5] by (1.1), (4.13), (4.14), (4.16), and (3.11), respectively. They comprise the set of formulas in the last column of the table in [5, appendix D]. Each of (1)–(5) is connected with a particular root system; details may be found in [4], [5].

In this paper we discuss character sum analogues of (1)–(5). They are obtained very roughly as follows. The factors $n!$ on the right are replaced by Gauss sums $G(N)$, where N is a character on GF(q). The factors on the left such as $(1-z)^n$ are replaced by $N(1-z)$, and "C.T." is replaced by a summation $q^{-n}\Sigma$ over $x_1,\cdots,x_n$, with each x_i running through the nonzero elements of GF(q).

Formula (1) is the well known "Dyson conjecture". A character sum analogue for $n=3$ is proved in [2, (5)]. It is

$$\text{(6)} \quad \sum_{x,y\neq 0} AC\left(\frac{1+x}{y}\right)BC\left(\frac{1+y}{x}\right)AB(y-x) = D(A,B,C)+D(A\phi,B\phi,C\phi),$$

with

$$\text{(7)} \qquad D(A,B,C) := \frac{q^2B(-1)G(ABC)}{G(A)G(B)G(C)},$$

provided that A^2, B^2, C^2, AB, AC, and BC are nontrivial. No such analogues of (1) have been proved or even conjectured for $n > 3$.

Formula (2) follows from Selberg's multivariable extension of Cauchy's beta integral formula. A character sum analogue of (2) for $n = 2$ is proved in [2, (4)]. It is

$$\text{(8)}\qquad \sum_{x,y} A(xy)B((1-x)(1-y))C^2(x-y) = R(A,B,C) + R(A,B,C\phi),$$

with

$$\text{(9)}\qquad R(A,B,C) := \frac{G(C^2)G(A)G(AC)G(B)G(BC)}{G(C)G(ABC)G(ABC^2)},$$

provided that ABC^2 and $(ABC)^2$ are nontrivial. The special case of (8) where $A = B = C^2 = \phi$ has been applied in graph theory [3]. Analogues of (2) for all $n > 2$ are given conjecturally in [2, (29)]. They are indirect in the sense that the character sums do not spring directly from the Laurent series in (2) but rather from that series after a change of variables ($x_1, \cdots, x_n$ are replaced by the elementary symmetric functions in new variables $X_1, \cdots, X_n$). No "direct" analogues of (2) for $n > 2$ appear to be known. For example, for $n = 3$, no evaluation in terms of Gauss sums has been proposed for the sum

$$\text{(10)}\qquad Q = \sum_{x,y,z} A(xyz)B((1-x)(1-y)(1-z))C^2((x-y)(x-z)(y-z)).$$

In the special case $1 \neq A = B = \bar{C}^2$, however, the sum Q can be evaluated; see the Appendix.

The main purpose of this paper is to prove character sum analogues of (3) and (4), in Theorems 1 and 2. We also give in Theorem 3 a character sum analogue of (5) in the case $n = 1$. No such direct analogue of (5) has been proved or conjectured for $n > 1$, but the conjectures in [2, (29)] may be viewed as indirect analogues.

2. Character sum analogues

We begin with an analogue of (3).

THEOREM 1. *If $AB \neq 1$ and $C^2 \neq 1$, then*

$$S := \sum_{t,u\neq 0} A((1-t)(1-\bar{t}))B((1-u)(1-\bar{u}))C((1-tu)(1-\bar{t}\bar{u}))$$

$$= F(A,B,C) + \phi(-1)F(A\phi, B\phi, C\phi),$$

where

$$F(A,B,C)=\frac{G(ABC)G(A^2)G(B^2)G(C^2)G(\bar{A}\bar{C})G(\bar{B}\bar{C})G(\bar{A}\bar{B})}{qG(A)G(B)G(C)}.$$

PROOF. Set

$$x=\frac{ut-t}{1-ut},\qquad y=\frac{u-1}{1-ut},$$

so

$$t=\frac{x}{y},\qquad u=\frac{1+y}{1+x}.$$

With this change of variables, we have

$$S=ABC(-1)\sum_{x,y}\bar{A}\bar{C}(xy)\bar{B}\bar{C}((1+x)(1+y))A^2B^2C^2(y-x). \tag{11}$$

By (8) and (11), since AB and C^2 are nontrivial,

$$S=ABC(-1)\{R(\bar{A}\bar{C},\bar{B}\bar{C},ABC\phi)+R(\bar{A}\bar{C},\bar{B}\bar{C},ABC)\}.$$

It remains to show that

$$F(A,B,C)=ABC(-1)R(\bar{A}\bar{C},\bar{B}\bar{C},ABC\phi). \tag{12}$$

By the definition (9), the right side of (12) equals

$$\frac{ABC(-1)G(A^2B^2C^2)G(\bar{A}\bar{C})G(B\phi)G(\bar{B}\bar{C})G(A\phi)}{G(ABC\phi)G(C\phi)G(AB)}$$

$$=\frac{\phi(-1)}{q^2}G(\bar{A}\bar{C})G(\bar{B}\bar{C})G(\bar{A}\bar{B})\frac{G(A^2B^2C^2)}{G(ABC\phi)}\{G(A\phi)G(B\phi)G(C\phi)\}.$$

It is well known [1, theorems 2 and 3] that for any character E,

$$G(E)G(E\phi)=\bar{E}(4)G(E^2)G(\phi). \tag{13}$$

Thus the right side of (12) equals

$$\frac{\phi(-1)}{q^2}G(\bar{A}\bar{C})G(\bar{B}\bar{C})G(\bar{A}\bar{B})\frac{ABC(4)G(ABC)}{G(\phi)}\left\{\frac{G(A^2)G(B^2)G(C^2)G^3(\phi)}{ABC(4)G(A)G(B)G(C)}\right\}$$

$$=F(ABC).$$

REMARK. If $C^2=1$, a direct argument shows that

$$S=J(\bar{A},A^2)J(\bar{B},B^2)-J(\bar{A}\bar{B},A^2B^2).$$

We can similarly evaluate S if $A^2=1$ or $B^2=1$, since, as is easily seen from the definition, S is symmetric in A, B, C. Now suppose that A^2, B^2, and C^2 are nontrivial. Then not all of AB, BC, AC can be trivial, and by symmetry, it may be supposed that $AB \neq 1$. Then Theorem 1 can be applied to evaluate S.

We next prove an analogue of (4).

THEOREM 2. *If $A^2 \neq 1$ and $A^2BC \neq 1$, then*

$$T:=\sum_{x,y\neq 0} A((1-x)(1-\bar{x})(1-y)(1-\bar{y}))B((1-x\bar{y})(1-y\bar{x}))C((1-xy)(1-\bar{x}\bar{y}))$$

$$=H(A,B,C)+\phi(-1)H(A\phi,B\phi,C)+\phi(-1)H(A\phi,B,C\phi)+H(A,B\phi,C\phi),$$

where

$$H(A,B,C)=\frac{A(-1)G(A^2)G(B^2)G(C^2)G(A^2BC)G(\bar{A}\bar{B}\bar{C})G(\bar{A}\bar{B})G(\bar{A}\bar{C})}{qG(A)G(B)G(C)}.$$

PROOF. Set

$$r=\frac{x+1}{x-1}, \qquad s=\frac{y+1}{y-1},$$

so

$$x=\frac{r+1}{r-1}, \qquad y=\frac{s+1}{s-1}.$$

With this change of variables, we have

$$T=A^2BC(-4)\sum_{r,s}\bar{A}\bar{B}\bar{C}((r^2-1)(s^2-1))B^2(s-r)C^2(s+r).$$

Now set $v=rs+1$, $u=(r+s)/2$. Then

$$T=A^2BC(-4)\sum_{u,v}\bar{A}\bar{B}\bar{C}(v^2-4u^2)B(4(u^2+1-v))C(4u^2)\{1+\phi(u^2+1-v)\}$$

$$=S(A,B,C)+S(A,B\phi,C\phi),$$

where

$$S(A,B,C)=BC(-1)A^2B^2C^2(4)\sum_{u,v}\bar{A}\bar{B}\bar{C}(v^2-4u^2)C(u^2)B\phi(u^2+1-v).$$

Observe that $S(A,B,C)=W(A,B,C)+\phi(-1)W(A\phi,B,C\phi)$, where

$$(14)\quad W(A,B,C)=BC(-1)A^2B^2C^2(4)\sum_{u,v}\bar{A}\bar{B}\bar{C}(v^2-4u)C\phi(u)B\phi(u+1-v).$$

It remains to prove that $W(A,B,C)=H(A,B,C)$.

By (14) and [2, (19)], since A^2 and A^2BC are nontrivial,

$$W(A,B,C)=BC(-1)A^2B^2C^2(4)R(C\phi,B\phi,\bar{A}\bar{B}\bar{C}\phi)$$

$$=\frac{BC(-1)A^2B^2C^2(4)G(\bar{A}^2\bar{B}^2\bar{C}^2)G(C\phi)G(\bar{A}\bar{B})G(B\phi)G(\bar{A}\bar{C})}{G(\bar{A}\bar{B}\bar{C})G(\bar{A}\phi)G(\bar{A}^2\bar{B}\bar{C})}$$

$$=\frac{A\phi(-1)}{q^2}A^2B^2C^2(4)G(A^2BC)G(\bar{A}\bar{B})G(\bar{A}\bar{C})$$

$$\times\frac{G(\bar{A}^2\bar{B}^2\bar{C}^2)}{G(\bar{A}\bar{B}\bar{C}\phi)}G(A\phi)G(B\phi)G(C\phi).$$

Applying (13) as in the proof of Theorem 1, we see that $W(A,B,C)=H(A,B,C)$.

REMARK. If $A^2=1$ or $A^2BC=1$, it is not hard to evaluate T; see the remark at the end of [2, §6].

We next prove an analogue of (5) for $n=1$.

THEOREM 3. *If $AC^2\neq 1$, then*

$$Y:=\sum_{x\neq 0}A((1-x)(1-\bar{x}))C((1-x^2)(1-\bar{x}^2))$$

$$=M(A,C)+\phi(-1)M(A,C\phi),$$

where

$$M(A,C)=\frac{qG(A^2C^2)G(C^2)}{G(AC)G(C)G(AC^2)}.$$

PROOF. $Y=AC(-1)\sum_{x\neq 0}A((x+\bar{x})-2)C((x+\bar{x})^2-4)$

$$=AC(-1)\sum_{u}A(u-2)C(u^2-4)\{1+\phi(u^2-4)\}$$

$$=L(A,C)+\phi(-1)L(A,C\phi),$$

where

$$L(A,C)=AC(-1)\sum_{u}AC\phi(u-2)C\phi(u+2).$$

Observe that

$$L(A,C)=\phi(-1)\sum_u AC\phi(4-u)C\phi(u)=\phi(-1)AC^2(4)J(AC\phi,C\phi).$$

Since $AC^2\neq 1$,

$$L(A,C)=\phi(-1)AC^2(4)G(AC\phi)G(C\phi)/G(AC^2).$$

Applying (13), we have $L(A,C)=M(A,C)$ and the result follows.

REMARK. If $AC^2=1$, then

$$Y=\phi(-1)J(\bar{C}\phi,C\phi)+J(\bar{C},C),$$

so

$$Y=\begin{cases} C(-1)(q-3), & \text{if } C^2=1, \\ -2C(-1), & \text{if } C^2\neq 1. \end{cases}$$

A character sum corresponding to the left side of (5) for $n=2$ is

$$V=\sum_{x,y\neq 0} A((1-x)(1-\bar{x})(1-y)(1-\bar{y}))C((1-x^2)(1-\bar{x}^2)(1-y^2)(1-\bar{y}^2))\cdot$$
$$B((1-xy)(1-\bar{x}\bar{y})(1-x\bar{y})(1-\bar{x}y)).$$

There is no known formula for V in terms of Gauss sums. However, if $C^2=1$, then V can be easily evaluated by applying Theorem 2 (with $B=C$) together with Theorem 3.

Appendix

We evaluate the sum Q in (10) in the special case $1\neq A=B=\bar{C}^2$. In this case,

$$Q=\sum_{x,y,z} A(xyz(1-x)(1-y)(1-z))\bar{A}((x-y)(x-z)(y-z))$$

$$=\sum_{x,y,z\neq 0} A((\bar{x}-1)(\bar{y}-1)(\bar{z}-1))\bar{A}((\bar{x}-\bar{y})(\bar{x}-\bar{z})(\bar{y}-\bar{z}))$$

$$=\sum_{x,y,z\neq 0} A((x-1)(y-1)(z-1))\bar{A}((x-y)(x-z)(y-z))$$

$$=\sum_{x,y,z\neq -1} A(xyz)\bar{A}((x-y)(x-z)(y-z))$$

$$= \sum_{x,y,z} A(xyz)\bar{A}((x-y)(x-z)(y-z))$$

$$-3\sum_{x,y} A(-xy)\bar{A}((x-y)(x+1)(y+1)).$$

Replacing x, y by xz, yz in the first sum just above, we see that

$$Q = \sum_{z \neq 0} Q_1 - 3Q_1 = (q-4)Q_1,$$

where

$$Q_1 = \sum_{x,y} A(xy)\bar{A}((1-x)(1-y)(x-y)).$$

By (8),

$$Q_1 = \frac{q^2 C(-1)G(C^3)}{G^3(C)} + \frac{q^2 C\phi(-1)G(C^3\phi)}{G^3(C\phi)}.$$

Thus

$$Q = (q-4)q^2 C(-1)\left\{\frac{G(C^3)}{G^3(C)} + \frac{\phi(-1)G(C^3\phi)}{G^3(C\phi)}\right\}.$$

EXAMPLE. If $q = p = a^2 + b^2$ with b even, then

$$Q = \sum_{x,y,z} \phi(xyz(1-x)(1-y)(1-z)(x-y)(x-z)(y-z)) = 2(p-4)(a^2-b^2).$$

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